

A Note on Theorem of Unequal Pair of Lunes

Gurajada Suryanarayana Murty

Abstract: It is shown that the sum of areas of pair of lunes, bounded by arcs of the semi-circles upon sides of a right-angled triangle inscribed in a circle, is equal to the area of the triangle.

Key words: Lune, Mathematical modeling

AMS Subject Classification: 00A30; 00A71; 03A05; 97D20

1. Introduction

A Lune is a geometrical figure formed on a plane by two intersecting arcs of circles on a plane, or on sphere, by two great circles. Hippocrates of Chios (c 430 B.C.) discovered a theorem that the area of a lune bounded by an arc of $\pi/2$ radians and by a semi-circle upon its chord, is equal to the area of the triangle formed by the corresponding chord with the center of the circle as its apex.[1]

Since this result is found to be useful in other contexts, we will review the theorem of Hippocrates. The Fig. 1 illustrates the geometrical details of a pair of equal lunes.

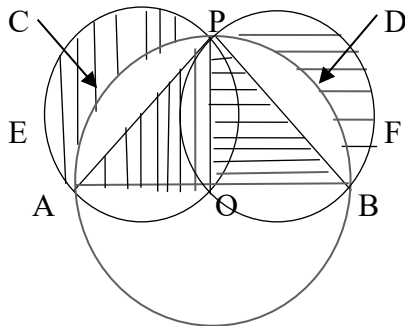


Fig. 1. The discovery of Hippocrates of Chio

In Fig 1, AOB is diameter of a circle with center at O and OP is perpendicular to AOB. AP and PB are chords which subtend an angle $\pi/2$ at the center and the $\angle POB = \angle POA = \pi/2$. Therefore, the small arcs ACP and PDB subtend an angle $\pi/2$ radians at the center. With AP and PB as diameters, two circles are drawn forming two lunes shown by the areas shaded by lines drawn vertically and horizontally. The triangles with AOP and POB are shaded similarly.

The area of the lune formed by the small arc ACP is equal to the area of the triangle AOP. Similarly, the same result holds for the other lune and corresponding triangle. This is the theorem of Hippocrates of Chio.

The relation between the lunes and the triangles when the angle subtended at the center is not equal to $\pi/2$, is not obvious. It does not seem to have been investigated as far as my knowledge goes.

It is shown in this Note that the sum of areas of pair of lunes bounded by arcs of the semi-circles upon sides of a right-angled triangle inscribed in a circle is equal to the area of the triangle.

2. Theorem

The Fig. 2 shows the construction of two lunes that subtend supplementary angles at the center.

Let AOB be the diameter of the circle with the center at O. The arc AECFB is one semi-circle. Join AC and CB to form the triangle ACB. Let the angle COB be equal to θ , and therefore the angle COA is equal to $\pi - \theta$. With AC and CB as diameters draw the semi-circles ADC and CGB respectively to form pair of lunes ADCEA and CGBFC. See Fig 2.

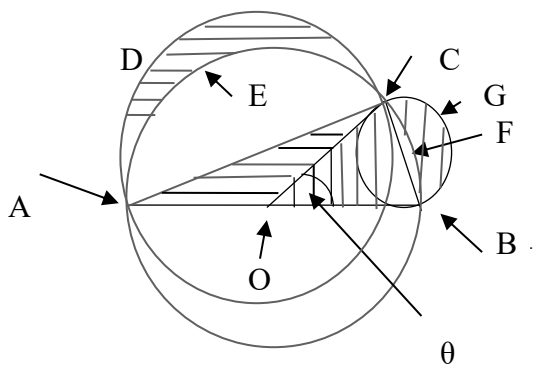


Fig 2. Construction of a pair of unequal lunes.
(Angle θ , measured anti-clockwise)

Theorem: The sum of the areas of lunes ADCEA and CGBFC is equal to $\sin \theta$, which is the area of the triangle ACB.

Proof: Let us first derive an expression for the area of lune bounded by ADCEA. Assume for simplicity that $OA = 1$. Following the rules of Euclidean geometry, the length of the chord AC is $2 [\sin \{(\pi-\theta)/2\}]$. Since $OA=1$, it follows that the area $S1$ of the semi-circle ADC is,

$$S1 = (\pi/2) \sin^2 [\{(\pi-\theta)/2\}] \quad (1)$$

Further, the area $S2$ enclosed by the arc AEC and the radii OA and OC is

$$S2 = (\pi-\theta)/2 \quad (2)$$

The area $S3$ of the triangle AOC is

$$\begin{aligned} S3 &= \sin [\{(\pi-\theta)/2\}] \times \cos [\{(\pi-\theta)/2\}] \\ &= (1/2) \sin [2 \{(\pi-\theta)/2\}] = (1/2) \sin [(\pi-\theta)] \\ &= (1/2) \sin \theta. \end{aligned} \quad (3)$$

Therefore, the area $S4$ of the lune ADCEA is given by

$$\begin{aligned} S4 &\equiv S1 - (S2 - S3) \\ &= (\pi/2) \sin^2 [\{(\pi-\theta)/2\}] + (1/2) \sin \theta - (\pi-\theta)/2 \\ &= (\pi/2) \cos^2 [\theta/2] + (1/2) \sin \theta - (\pi-\theta)/2. \end{aligned} \quad (4)$$

The area $S5$ of the lune CGBFC is obtained from the above equation for $S4$ by changing θ to $\pi-\theta$.

Therefore

$$S5 = (\pi/2) \sin^2 [\theta/2] + (1/2) \sin \theta - \theta/2. \quad (5)$$

And, finally it can be verified that the sum of areas of the two lunes, $S6$, is

$$\begin{aligned} S6 &= S4 + S5 \\ &= \sin \theta, \end{aligned} \quad (6)$$

which is the area of the triangle ACB, since the triangles COB and AOC have equal bases and heights. (Q.E.D).

It is interesting to examine the ratio of the area of lune ADCEA, and that of the triangle AOC. It can be shown using Eqs. (3) and (4) that this ratio, denoted as R1, is given by

$$R1 = 1 + (\pi/2) \cot (\theta/2) - (\pi-\theta)/ \sin \theta. \quad (7)$$

If $\theta=\pi/2$, we obtain $R1 = 1$ which agrees with the result of Hippocrates. It can be verified that R1 tends to 2 or zero when θ tends to 0 or π .

Similarly, it can be shown, that the ratio, R2, of the area of lune CGBFC and the area of triangle OBC is given by

$$R2 = 1 + (\pi/2) \tan (\theta/2) - \theta/\sin \theta. \quad (8)$$

It can also be shown that $R1 + R2 \equiv 2$. (9)

The variation of R1, R2, S4. and S5 with the angle θ , $0 \leq \theta \leq \pi$, is shown in the Fig 3.

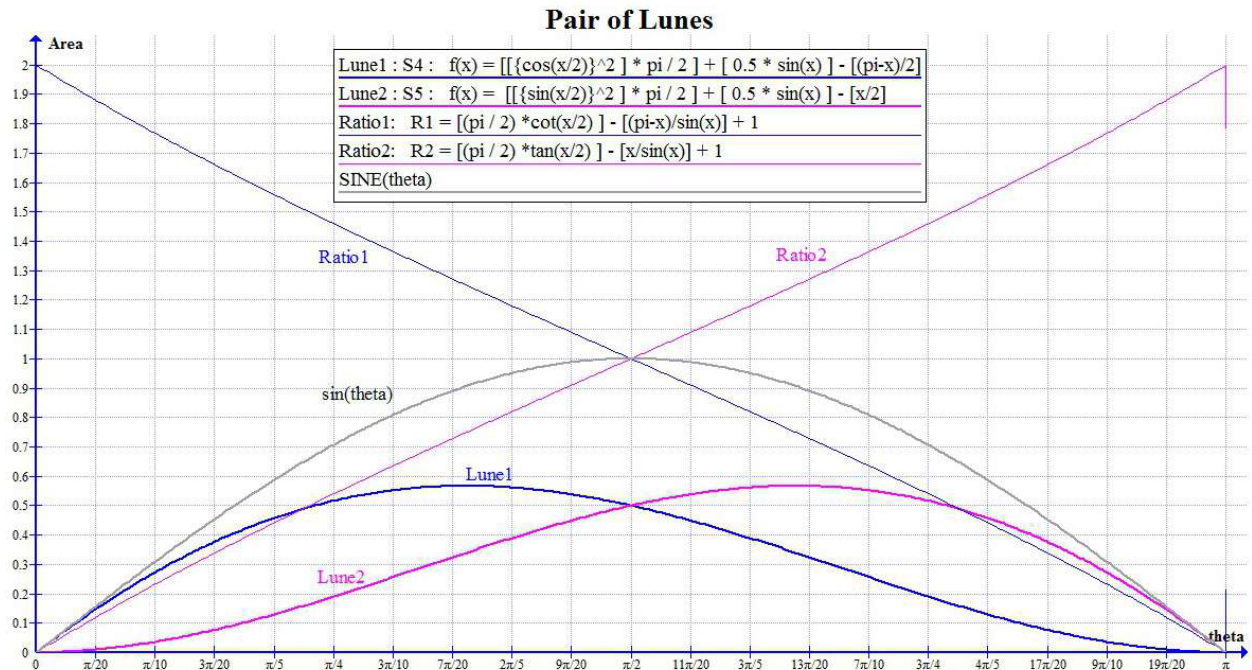


Fig 3: Graphs of S4, S5 ; R1, R2 and Sin(theta)

3. Conclusion

This theorem is an example in which the interior area of a circle is related to its exterior area.

Pythagoras theorem was useful in several contexts, but the theorem of Hippocrates of Chios (c 430 B.C.) was not found to have applications. This problem was recently studied in an attempt to develop a mathematical model of a ‘Self Conscious Unit’ in which there would be two domains one of which is an ‘interior’ domain and the other is an ‘exterior’ domain. In that context, the theorem of Hippocrates was found to be useful.

The geometry of a Lune involves a confluence of three concepts: (a) a circle; (b) a chord of the circle; and (c) a semi-circle on the chord. These three notions are purely geometrical notions. The ‘special’ interest in the relation between the area of the lune and the area of the triangle formed by the chord at the center of the circle, is their ‘equality’.

At the time of Hippocrates of Chios (c 430 B.C.), the transcendental functions were not known. We are now sure of the equation $\exp \{i(\pi/2)\}=1$. The relation between real numbers and transcendental numbers has enabled us to bridge the gulf between mathematics and metaphysics.

In a book entitled *Paratattvaganithadarsanam*, [2] which was reviewed in the *Journal of Indian Academy of Mathematics* [3], a brief discussion of unequal lunes was given, and the results were used to understand the enigmatic Upanishadic statements of Pair of Birds seated on a Branch of a Tree [4].

Acknowledgement

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References

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[4] The M (III. 1.1 and III.1.2).

Address: (at the time of this writing, ~Aug 2017)

G.S.Murty

Plot # 18, Vertex Lake View

Nizampet,

Hyderabad 500 090

Email: gurajadamurty@yahoo.com

Latest Address (Feb 2023)

c/o Mrs. G. Satyakumari,

2-2-1109/3/5 Sai Tulip, Apt B3

Ganga Bowli, Bagh Amberpet

Hyderabad – 500 The013

Email: adityagurajada@yahoo.com

Update (Feb 2023): This paper was published by Dr. Murty in some journal / publication, but I fail to recall where it was published. It may have been published in Journal of Indian Academy of Mathematics, Indore, but I could not locate a final printed version of this paper. – Aditya.