# AUM – On the Structure of Space-Time:
# A Unifying Model for Classical, Relativistic and Quantum Space-Time


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Abstract

We propose a kinematical model of space-time encompassing classical, relativistic and quantum concepts assuming that Planck's length $\sqrt{(Gh/c^3)}$ has a role to play in the relation between space and time variables, such as in Minkowski's space-time relation.

This paper attempts the scaling of space to reveal a structure in the Space-Time complex. Scaling of space leads to a new Quantum Space-Time. In the proposed Quantum Space-Time, in accordance with Quantum Physics, absolute rest is not possible since speed turns out to be a singularity of the Space-Time complex. The proposed Space-Time structure also turns out to depend on speed. We show that the Space-Time structure may approach the Classical Newtonian regime, or the Relativistic Minkowskian regime. This transition resembles thermodynamic phase-transitions, as an analogy.

The proposed model re-examines the assumption in the Theory of Special Relativity (TSR) of the constancy of speed of light, and validates it to be a requirement of TSR, thereby removing the ad-hoc nature of this assumption. This model also reveals inherent indeterminacy between position and speed in the space-time structure, thereby reconfirming the Uncertainty Principle of Quantum mechanics.

Our model reveals a space-time structure which is Quantum in nature, thus unifying Classical and Relativistic space-time structures in its fold.

Keywords: Planck's length, Space-time, Relativity, Kinematics, Singularities, Quantum Space-time

AMS Classification: 03A; 14H; 32S;70B; 83A.

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# On the Structure of Space-Time:
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## 1. Introduction

Recent trends in theoretical physics can be grouped under two major headings. One is the study of the macroscopic material world governed by the gravitational constant, G, introduced by Newton, and the other depends on the Planck's constant, h, dealing with the behavior of matter at microscopic levels. In between these two, the electromagnetic properties of matter and the relativistic properties of space-time play the role of a link through another constant, namely, the speed of light, c, in vacuum. In spite of the vast developments in the experimental and theoretical research, these fields are not unified conceptually. The difficulties in the unification of quantum mechanics with relativity principle arise out of the inadequacy in the definition of the kinematical features of space-time. The source of conflict is traced ultimately to the very different roles of time in quantum mechanics and general relativity [20]. Relativity principle assumes that the speed of light should be an absolute constant for all inertial frames of an observer. Quantum mechanics on the other hand shows how to make unique theoretical predictions only when the role of time is separated from the rest of the variables and is treated as a c-number [16]. Even the validity of Lorentz transformation itself is being questioned in a domain comparable to Planck's length [20]. Models of space-time in which time is not a continuous variable have also been studied with the belief that there may be fundamental limiting length underlying the physical world [12][21]. Wilzcek suggests[1] [9] that it may be necessary to include fundamental constants into our description of nature.

[1] Source [9] "In the correspondence between equations and reality, the values of those parameters are not determined conceptually, but must be provided empirically, by measurements. In this way, we introduce fundamental constants into our description of Nature."

In the contemporary developments of our notions of space-time after Newton, the principle of Special Relativity occupied the central place. It was however formulated on the basis of the assumption of the constancy and the absolute nature of speed of light in vacuum. After the TSR was generalized by Einstein [6] to Theory of General Relativity, (TGR) there were extensive debates among theoretical physicists regarding the status of space and time, particularly after quantum nature of light and matter at microscopic levels was discovered. In his formulation of relativity principle, Einstein said: "In developing any physical theory of space and time, processes of some kind are required which enable relations to be established between different places and times. It is immaterial what kind of process one chooses concerning which we know something certain. This holds for the propagation of light in *vacuo* to a higher degree than for any other process." [*Meaning of Relativity*. Princeton 1950 pp. 28-29]. While applying the principle of relativity to frames of reference in uniform motion, two concepts remained unclear: (1) the concept of an 'event', and (2) the assumption of unlimited precision in the notion of simultaneity of two events.

Lande [17] indicated a method to understand "non-relativistic quantum theory based on postulates of thermodynamic *continuity* and mechanical *conjugacy*". He also said that "possession of the formal structure of quantum theory does not solve the greater riddle of the constitution of matter itself with its various elementary particles and their interaction, transformation, creation, annihilation etc. It may be that a relativistic quantum theory together with the introduction of an elementary length will solve these riddles." It was said by Lande [17] that the easiest avenue of approach to quantum theory has yet to be determined though, and as stated by Whittaker [23]: "There are many roads that lead to quantum theory". In a similar tone, it can be said today that many roads exist to incorporate Planck's length in our notions of space and time, but the easiest avenue of approach is yet to be obtained [24].

The rest of the paper is organized as follows. In Sec. 2, we present the motivation behind this work that lead to a re-examination of Minkowski's space-time equation to incorporate Planck's length. We also present the reasoning behind this change and the consequences of such a change. Section 3 reiterates the summary of a Kinematical model of space-time which has been previously been examined by Murty **Error! Reference**

**source not found.**. We propose small transformations to Minkowski's equation and study the implications of these changes. Section 4 discusses the parametric and analytical solutions to the revised space-time equation, treating the modified equation as a differential equation. Section 5 discusses the interpretation and physical implications of the solutions of the modified differential space-time equation. Section 6 shows how the modified space-time structure can be generalized in terms of a parametric variable, and combinations of other fundamental constants. In section 7 we present the concept of Quantum space-time as embodied in our proposed model, and regimes of space-time defined as a function of speed of light. Section 8 presents some discussion points of the mathematical results obtained from this new model and their physical implications. We finally present our conclusions and new contributions of this model in Section 9.

## 2. Motivation

The stipulation in TSR that the speed of light is an absolute constant and that Space and Time must be invariably related by an equation known as Minkowski equation is generally accepted. This equation is written as,

$$x^2 - c^2 t^2 = 0$$

where the variables 'x' and 't' represent 'space' and 'time', respectively, and $c = dx/dt$ is the speed of light in vacuum, which is accepted to be an absolute constant.

Physically, TSR equates the *geometrical Euclidean length*, x, with the *experimental length*, (ct), measured using the speed of light. The geometrical knowledge of a line is obtained by, say, the use of a measuring scale. This implies that one is able to measure both ends of the line simultaneously, which tacitly assumes an infinite speed of light. The experimental length obtained needs the knowledge of speed of light, known to be finite, and the time interval for light to cover that distance. Here we are assuming [that] the geometrical length is identical with the experimental length. From a practical point of view, this may be accepted as an empirical truth because it is also assumed that the speed of light is known accurately and time measurement is also accurate.

However, there are some hitherto unexplored issues, as follows:

**a)** Philosophically, there seemed to be an inconsistent treatment of the metric being compared, i.e. the Euclidean length measured assuming an infinite speed of light, v/s the experimental length measured using light with a finite speed.

**b)** Also, howsoever accurate, the speed of light, c, is still a measured entity, which may be subject[2] to some slight measurement error.

Both these issues lead us to consider the possibility that the difference between these two "length" values, or more accurately the difference of the computed area units as shown above, need not necessarily be 0. Hence, we considered the possibility of changing the right-hand side (r.h.s) of the Minkowski equation to be a non-zero value, and the consequences of such a transformation.

Among the known physical constants related to space-time concepts, two stand out prominently: Newton's Gravitational constant, G, and Planck's constant, h. Speaking heuristically, one could say that if Newton's constant were zero, there would have been no solar system. And, if Planck's constant were zero there would have been no stable atoms. The units of their product, Gh, (*G* multiplied by *h*) turns out to be, $L^5/T^3$, which involves only properties of space and time. This suggests that a kinematical model of space-time could be defined in terms of these constants.

It has been recognized from the study of nature that the combination of the constants like $Gh/c^3$ defines an area and $\sqrt{(Gh/c^3)}$ is called 'Planck's length [9][24]. The role of Planck's length has been studied previously [24], but not in a manner presented in this work. Square of the Planck's length; i.e., $Gh/c^3$, defines an area, referred here loosely as Planck's area. The current interpretation is that Planck's length is the shortest meaningful length, the limiting distance below which the very notions of space and length cease to exist. Planck's length is the length scale at which the structure of space-time becomes dominated by quantum effects, and it is impossible to determine the difference between two locations less than one Planck length apart. We considered extending Planck's length to Planck's area, whereby there cannot be an 'area' which is arbitrarily smaller than Planck's area. Consequently, we interpreted Planck's area to

[2]Einstein noticed this difficulty and said, "We shall not here discuss the inexactitude which lurks in the concept of simultaneity of two events at approximately same place which can be removed by an abstraction."

mean the smallest realizable area in the material universe, which is extremely small but certainly a non-zero value, roughly of the order of $10^{-70}$ sq. m. It means that in a material universe there cannot be an ‘area’ which is arbitrarily small. By allowing a non-zero value on the right-hand side converts the Minkowski equation to a hyperbolic form, which yields a mathematical identity. This property lends credence to the approach attempted.

Thus, we examined whether these constants G and h, in the form of Planck’s area, have any role to play in Space-Time.

## 3. Kinematical Model of Space-Time

Historically a model of space-time has been continually evolving, for example, from Galilean space-time to Minkowski space-time. Now, in light of the above discussion, we ask the following question: Which of the following two theoretical models of the space-time should be accepted?

Model I: = C(G,h),

or,

Model II: = D(Gh),

where, apart from any other physical constants, C(G,0) ≡ A(G) stands for a ‘Theory-A’ depending only on ‘G’; and C(0, h) ≡ B(h) stands for ‘Theory-B’ depending only on ‘h’. C(G, h) stands for ‘Theory-C’, unknown at present, which depends on G and h. D(Gh) stands for ‘Theory-D’, which depends on the product Gh, also unknown at present. Theory-D is such that D(Gh) tends to A(G) or B(h) under clearly stated conditions which are physically acceptable.

In the Model I, the choices of either h=0 or G=0 have a meaning to us. Since we found that we could successfully develop macroscopic models A(G) as Newton did for Classical Physics, and models of microscopic universe B(h) as Planck did for Non-Relativistic Quantum Physics (in a space-time we temporarily label as Planck’s space-time), it seems that Model I should be accepted and that Model II need not be considered at all. But if we can show that there is a Model II which encompasses Model I as a special case, then Model II should be preferred. In view of the dimensionality of the product Gh, which is purely defined in terms of space and time dimensions, we propose a Model II i.e. D(Gh) that fulfills the kinematical requirement that space-time model involves only space and time variables and their derivatives.

An ‘element’ of philosophical expectation exists in every step taken to formulate a mathematical model of physical truth. The foundations of current theoretical physics, laid by Galileo and Newton, were based on the assumption that our concepts of space and time were absolute. A revision of these notions was made by the advent of the Theory of Special Relativity (TSR) in the beginning of last century, which introduced the notion that the speed of light must be accepted to be an absolute constant, thereby unifying space and time into a space-time complex. The classical view of space and time was discarded in favor of a space-time complex.

**Original equation – Minkowski Equation**:

An equation relating space and time, named after the mathematician Minkowski, came into existence, which is written as,

$$x^2 - c^2\, t^2 = 0 \qquad (1)$$

where the variables ‘x’ and ‘t’ represent ‘space’ and ‘time’, respectively, and c = dx/dt is the speed of light in vacuum, which is accepted to be an absolute constant, just like the gravitational constant, G, and Planck’s constant, h.

Eq. (1) is a kinematical condition on space and time variables and is the basis of Theory of Special Relativity (TSR), introduced by Einstein [7]. TSR states that the laws of physical phenomena should be the same in every inertial frame, that is, in a frame at rest (rest frame) or in uniform motion.

In light of the dimensionality of the product Gh – and a preference for the Model II of the form D(Gh) – we propose a simple generalization of the kinematical space-time relation of relativity principle in the following manner [18]:

- **First Change:** ***Scaling of Space***

We propose a new model by modifying the right-hand side (r.h.s) of the Eq. (1) to be a non-zero value, as shown below:

$$x^2 - c^2t^2 = \lambda^* \,( Gh / c^3) \qquad (2)$$

where $\lambda^*$ is some non-dimensional factor[3]. In this modified Eq. (2), $\lambda^*$ is provided as a mechanism to consider the role of other fundamental constants in the space-time structure.

Mathematically, Eq. (2) can be considered as scaling of space, is dimensionally consistent and the r.h.s term, Planck's area, is practically very small. Loosely speaking, in the limit when $Gh \rightarrow 0$, we can consider Eq. (2) to be identical to Eq. (1).

When the factor $\lambda^* = 0$, the Eq. (2) reduces to the relation given by Eq. (1). Rearranging terms, Eq. (2) can also be written as

$$c^3 (x^2 - c^2t^2) = \lambda^* Gh \tag{2a}$$

We can always modify Eq. (2a) so that $\lambda^*$ in the right-hand side of the equation has no space or time variable or their derivatives. For example, let's assume that $\lambda^*$ is the fine structure constant ($e^2/hc$), where 'e' is the unit of electronic charge and h is the Planck's constant. As this constant happens to involve a factor, c, we can still define another constant $\lambda$ such that

$$\lambda^* = \lambda/c$$

where, $\lambda = e^2/h$.

Eq. (2a) then becomes:

$$c^4 (x^2 - c^2t^2) = \lambda Gh \tag{2b}$$

*Thus, we must always choose $\lambda$ such that it is independent of space and time variables or their derivatives.*

Eq. (2b) can be generalized to:

$$c^n ( x^2 - c^2t^2) = \lambda Gh \tag{2c}$$

where, n=4, with this choice of $\lambda^*$.

- **Second Change: *Interpretation of Eq. 2(c) as a differential equation***

[3] The role played by the non-dimensional factor $\lambda^*$is discussed further in this paper in section 6.

We can view the Eqns. (2), (2a), (2b) and (2c) in two different ways. One view, which is the traditional view, is that it is an algebraic equation in variable c, where c is a constant, the speed of light in vacuum.

We propose a new interpretation of these equations, which is as follows. If we consider c to not be the constant speed of light but re-defined as *any speed*, dx/dt, then Eq. (2c) may now be written as follows:

$$c^n (x^2 - c^2t^2) = \lambda Gh, \quad \ldots \text{ where } c = dx/dt \qquad (3)$$

The left-hand side (l.h.s) of Eq. (3) comprises only of kinematical variables of space x, time t, and their derivative (dx/dt), i.e. speed. The r.h.s is also comprised of space and time variables coming from a combination of natural constants. Thus, both sides of the equation remain dimensionally consistent.

Henceforth, speed of light in vacuum will be denoted by the symbol $c_0$ and the quantity $\sqrt{(Gh/c_0^3)}$ becomes Planck's length. **To distinguish between the constant speed of light in vacuum and any speed, throughout this paper here onwards, we will use $c_0$ for the constant speed of light in vacuum, and retain the symbol c to represent any speed, as the derivative dx/dt.**

Note, that Eq. (2c) and Eq. (3) have the same form but Eq. (2c) is an algebraic equation whereas Eq. (3) is a differential equation[4]. The interpretations are different.

With these changes to the form and interpretation, Eq. (2) becomes Eq. (3), where Eq. (3) is an inhomogeneous non-linear first order differential equation connecting space-time variables, 'x' and 't', and their derivative (c=dx/dt).

Eq. (3) can be explicitly rewritten as follows:

$$(dx/dt)^n \, (x^2 - (dx/dt)^2t^2) = \lambda Gh \qquad (3a)$$

In the following sections, we will present the solutions and interpretations of the Eq. (3).

[4] It would be instructive to read in this context what Einstein wrote, [8], while explaining his formulation of Theory of General Relativity: "*... Adhering to the continuum originates with me not in a prejudice, but arises out of the fact that I have been unable to think up anything organic to take its place. How is one to conserve four-dimensionality in essence (or near approximation) and (at the same time) surrender the continuum?*" Einstein indicated in these words his trust in the essential role of differential equations, whether they are linear or non-linear, in our understanding nature.

## 4. Mathematical Solutions of the Differential Equation

We recast Eq. (3) that defines kinematical space-time relations, as below:

$$(x^2 - c^2t^2) = \lambda Gh/c^n, \quad (3b)$$

where, $n \geq 1$. `

We will rewrite Eq. (3b) in terms of non-dimensional variables by choosing suitable units for length and time, defined below as:

Unit of length, $U_s = (|\lambda Gh| /c_0^n)^{1/2}$, (4a)

Unit of time, $U_t = (|\lambda Gh| /c_0^{n+2})^{1/2}$, (4b)

so that

$$x = U_s\psi, \quad (4c)$$

$$t = U_t\xi, \quad (4d)$$

and

$$c = dx/dt = c_0 (d\psi/d\xi). \quad (4e)$$

so that $(d\psi/d\xi) = (c/c_0)$.

In the definitions (4a) and (4b), according to the sign of λ, it is possible that the right-hand side of Eq. (3b) may be a positive or a negative quantity and hence the product '$\lambda Gh$' is shown by the modulus.

Eqns. (3b) and (4a) through (4e) now lead us to the following equations:

$$p^n [\psi^2 - p^2 \xi^2 ] = +1, \quad (4)$$

$$p^n [\psi^2 - p^2 \xi^2 ] = -1, \quad (5)$$

where

$$p = (d\psi/d\xi). \quad (6)$$

Both Eqns. (4) and (5) are $(n+2)^{th}$ degree first-order inhomogeneous differential equations.

Note that, although one can convert Eq. (4) to Eq. (5) by the transformations shown below, we will study their solutions separately to understand any physical ramifications of the solutions.

$$\psi \to \psi' = \psi \exp (i\pi/2) \quad (7)$$

$$\xi \to \xi' = \xi \exp (i\pi/2) \quad (8)$$

**Notation**: Hereafter, we shall distinguish the solutions of Eqns. (4) and (5) by suffixes 1 and 2 attached to the respective symbols.

In the following sub-sections, we present the solutions of Eq. (3) valid for integral values of n and also give numerical results for the specific cases of n=3 (which corresponds to the choice $\lambda^*= 1$) and for case n=4. In Sec. 6, we will discuss why only n=3 and n=4 are more interesting, and their physical significance.

### 4(a) Analytical solution of Eqns. (4) and (5)

It can be verified that Eqns. (4) and (5) have analytical solutions given by

$$\psi_1^{n+2} = +K\,\xi_1^{n}, \qquad (9)$$

$$\psi_2^{n+2} = -K\,\xi_2^{n}, \qquad (10)$$

where

$$K = [(n+2)^{(n+2)}] / [4\,(n+1)\,n^{n}]. \qquad (11)$$

The details of the solution of Eq. (4) are presented in Appendix A1(a).

For n=3, the variations of $\psi_1$ with $\xi_1$, and $\psi_2$ with $\xi_2$ are shown in Fig.1.

Fig1

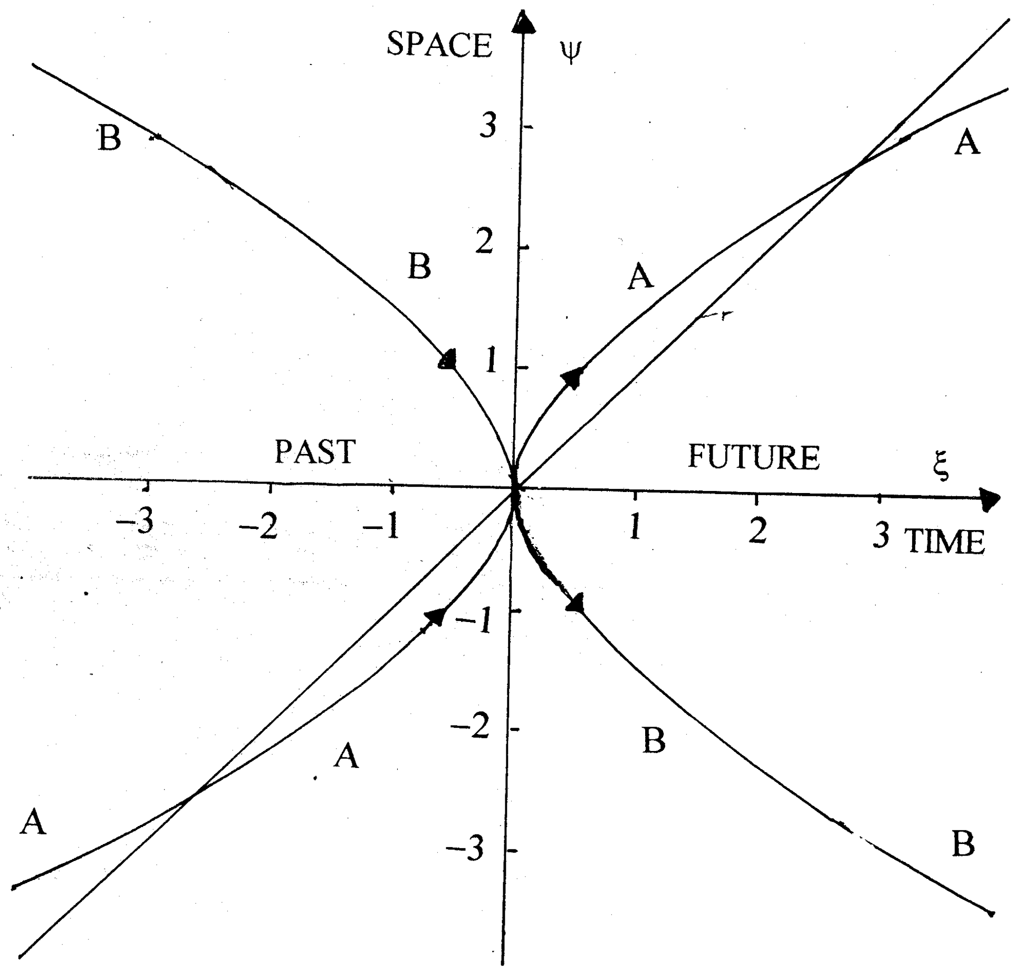


**Fig 1. Analytical solutions according to Eqns. (9) and (10), for n=3 (Source [18])**

In Fig. 1, for n=3, the variation of $\psi_1$ with $\xi_1$ is shown by the curve marked AAAA; the variation $\psi_2$ with $\xi_2$ is shown by the curve marked BBBB. By convention, the curve marked AAAA represents 'space-like' space-time, while the curve marked BBBB represents 'time-like' space-time. The straight line represents the invariant space-time relation according to relativity principle.

The space-time curves of Eqns. (9) and (10) appear like a cross section of sand-clock, but not like Minkowski's space-time cone. This does not mean the picture of Minkowski is to be discarded. It only means that space-time curve which is an analytical solution of the Eqns. (4) and (5) which passes through the origin of space-time coordinates does not have an asymptotic behavior, which is consistent with the TSR. As the curve approaches the origin, the tangent to these graphs, which represents speed, becomes higher than the speed of light. This is not permitted as per TSR.

**4(b) Parametric solution of Eq. (4)**

The parametric solutions of the Eqns. (4) and (5) are constructed based on a parameter, $\alpha$, which is a real variable ranging between $-\infty < \alpha < \infty$, as follows:

$$\psi_1(\alpha) = [\cosh(\alpha)] / [p_1(\alpha)]^{n/2}, \quad (12)$$

$$\xi_1(\alpha) = [\sinh(\alpha)] / [p_1(\alpha)]^{(n+2)/2}, \quad (13)$$

where $p_1(\alpha) = d\psi_1/d\xi_1$

$$p_1(\alpha) = [\{(n+1)\exp(-2\alpha) - 1\}]^{1/(n+1)}, \quad (14)$$

where, $-\infty < \alpha < [\,½ \ln(n+1)\,]$;

and $p_1(\alpha) = [\{1 - (n+1)\exp(-2\alpha)\}]^{1/(n+1)}$, (15)

where, $[\,½ \ln(n+1)\,] < \alpha < +\infty$.

The details of the solution for Eq. (15) are presented in Appendix A1(b).

Note, here we identify an $\alpha^*$ such that $p_1(\alpha)$ tends to 0 as $\alpha$ tends to $\alpha^*$, where

$\alpha^* = ½ \ln(n+1)$.

The significance of $\alpha^*$ will be discussed in a later section.

What is interesting to observe here is that the parametric solution in Eqns. (12) and (13) turn the parametric forms of the Differential equation, Eq. (4) into a trigonometric mathematical identity:

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1 \quad (16a)$$

and Eq. (5) into

$$\sinh^2(\alpha) - \cosh^2(\alpha) = -1 \quad (16b)$$

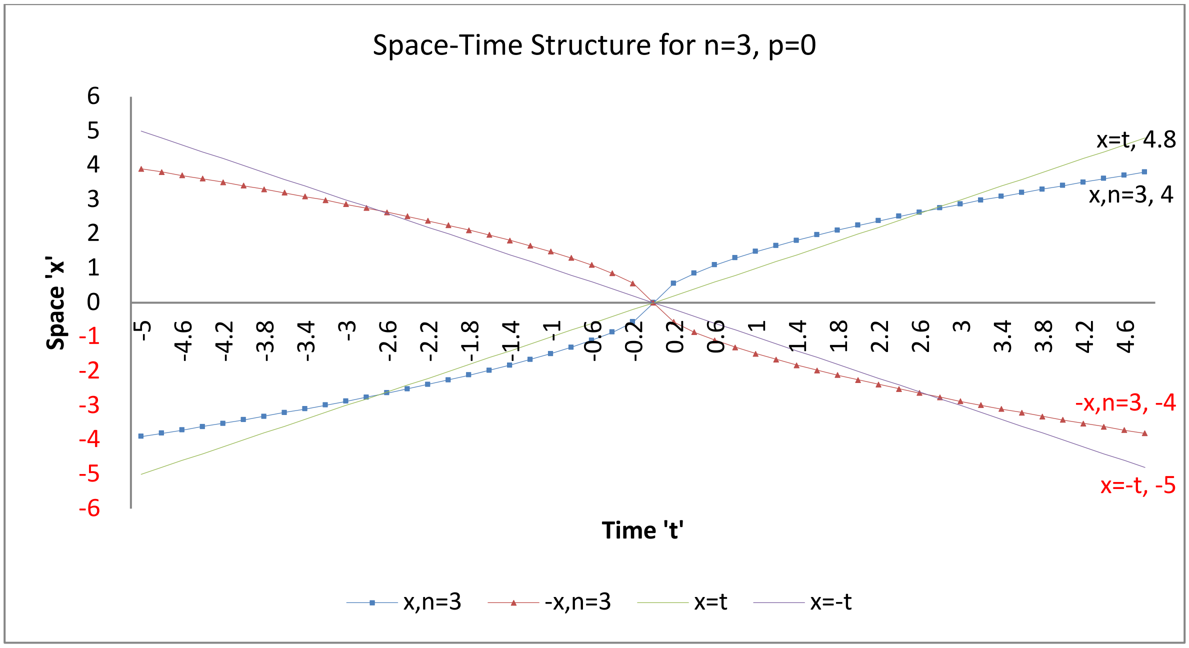


**Fig. 1(a) Analytical solution for n=3 of x v/s t, for speed p=0 (Same as Fig. (1))**

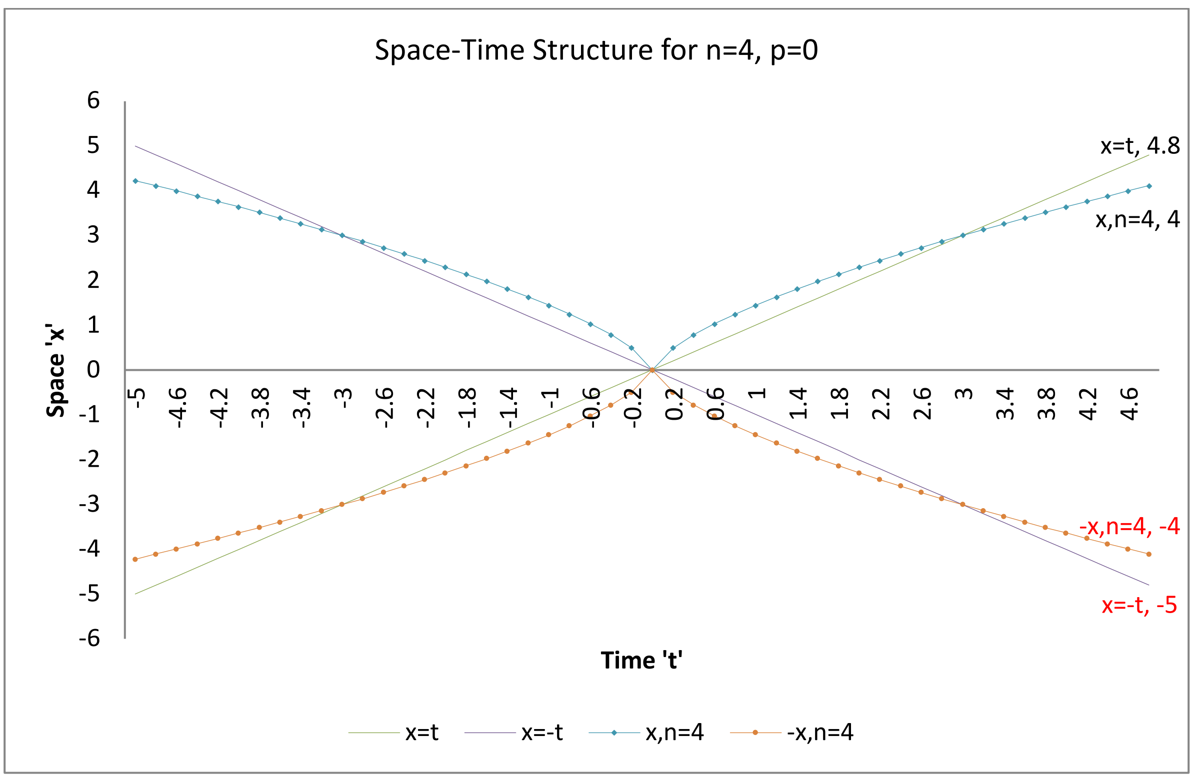


**Fig. 1(b) Analytical solution for n=4 of x v/s t, for speed p=0**

## 4(c) Parametric solution of Eq. (5)

The solutions are constructed based on a parameter, $\beta$, which is a real variable ranging between $-\infty < \beta < \infty$.

The parametric solution of Eq. (5) is given by

$$\psi_2 = [\sinh(\beta)] / [p_2(\beta)]^{(n/2)}, \quad (17)$$

$$\xi_2 = [\cosh(\beta)] / [p_2(\beta)]^{(n+2)/2}, \quad (18)$$

where,
$$p_2(\beta) = (d\psi_2 / d\xi_2)$$
$$= [1 + (n+1)\exp(-2\beta)]^{1/(n+1)}, \quad (19)$$

The details of the solution for Eq. (19) are similar to the solution for Eq. (15) as presented in Appendix A1 (b).

## 5. Interpreting the Solutions of Differential Eqns. (4) and (5)

We now interpret the solutions of the differential equations for the space-time structure to understand the physical implications of the solutions.

**Interpreting Solutions of Eq. (4)**

The parametric solutions of Eq. (4) are given by Eqns. (12) to (15) and the analytical solutions are given by the Eqns. (9) and (11).

**Interpreting the Parametric Solutions**

In the Eqns. (12) and (13), both $\psi_1(\alpha)$ and $\xi_1(\alpha)$ tend to infinity as $\alpha$ tends to the branch point, $\alpha^*$, (½ ln(n + 1)), where $p_1(\alpha) = 0$.

$$p_1(\alpha) \rightarrow 0, \ \alpha \rightarrow (½ \ln(n + 1)), \quad (20)$$

This is an interesting result by itself. It shows that the speed ‘p’ is a singularity of the space-time complex as in Eqns. (12) and (13). The speed p can only approach a value of zero asymptotically, as $\psi_1$ and $\xi_1$ tend to infinity. *The speed $p_1 = 0$ is not possible!* This is in accordance with the Quantum physics concepts where absolute rest is not permitted.

In Eqns. (12), (13) and (15), after $\alpha$ crosses the value ½ ln(n + 1) and tends to infinity in Eq. (15), the binomial expansion of Eq. (15) shows that:

$$p_1(\alpha) \rightarrow 1 - \exp(-2\alpha), \alpha \rightarrow \infty, \qquad (21)$$

Thus, as $\alpha$ approaches infinity, the speed $p1(\alpha)$ asymptotically approaches the speed of light in vacuum. This result is similar to the Minkowski formulation and conforms to the stipulations of the Special Theory of Relativity that no entity can be accelerated to the speed of light. This justifies that in special relativity the speed of light is a limit in this range of $\alpha$.

**Form-Equivalence of Parametric and Analytical Solutions**

Also, it is interesting to note the relation between the analytical Eq. (9) and the parametric Eqns. (12) and (13). From the Eqns. (12) and (13) we obtain

$$\psi_1^{n+2} = K_1 \xi_1^n \qquad (22)$$

Eq. (22) is similar to Eq. (9).

and, $$K_1 = [\cosh^{n+2}(\alpha)] / [\sinh^n(\alpha)] \qquad (23)$$

Eqns. (22) and (23) arrived from the parametric solution show the form-equivalence of the parametric solution to the analytic solution.

**Graphical representation of the Parametric Solution**

For n=3, the space-time curve is shown in Fig. 2 below. The variation of $p_1(\alpha)$ with $\alpha$ is shown in Fig. 3 below. Fig. 2 represents the space-time structure in which each point on this plot represents a space-variable and time-variable which are tied together with a unique value of $\alpha$. Each point on this plot corresponds to a speed-value.

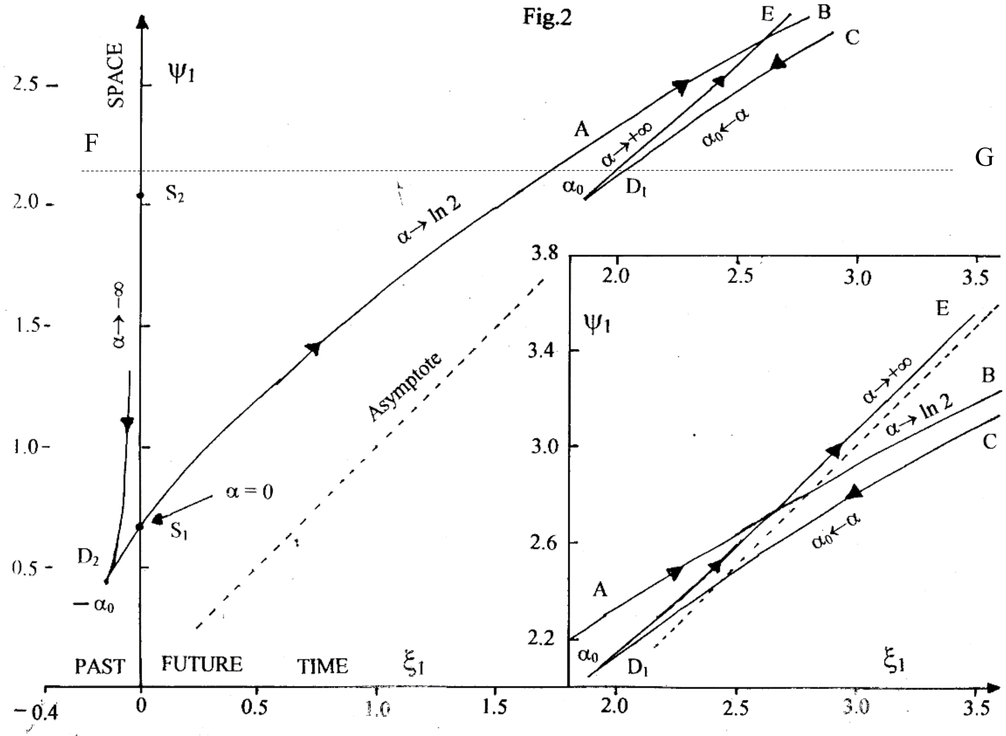


**Fig 2. Parametric solutions showing variation $\psi_1$ with $\xi_1$ according to Eqns. (12) and (13), for n =3. (Source [18])**

In Fig. 2, the parameter $\alpha$ varies from $-\infty$ to $+\infty$. The symbols $D_1$ and $D_2$ indicate respectively the cusps at $\alpha = \pm\alpha_0$, where $\psi_1$ and $\xi_1$ attain minimum values. As $\alpha\to\alpha^*$, ln 2, both $\psi_1$ and $\xi_1$ tend to infinity (arrow A to B) and space becomes independent of time. As $\alpha$ crosses the value ln 2, and attains the value $+\alpha_0$, both $\psi_1$ and $\xi_1$ decrease to reach the cusp at $D_1$ (arrow C to $D_1$). Thereafter, as $\alpha\to\infty$, both $\psi_1$ and $\xi_1\to\infty$ and reach the asymptote shown by the broken line (arrow $D_1$ to E). When $\xi_1 = 0$, $\psi_1 \neq 0$. In the whole range $-\infty < \alpha < +\infty$, $\psi_1 > 0$, but $\xi_1$ is positive or negative according to $\alpha$ is positive or negative. The points marked $S_1$ and $S_2$ are values of $\psi_1(0)$ and $\psi_1(\alpha_0)$, respectively. The inset shows the extension of a portion of top right region. The multiple values of $\psi_1$ at a given value of $\xi_1$ (and vice versa) are obvious.

In Fig. 2, the horizontal dashed-line FG represents a set of coordinates in this matrix with the same space coordinate. The meaning and implications of this line intersecting in more than one places with the graphs (A→B), ($D_1$→E), (C→$D_1$) shown on this space-time matrix will be discussed in a future section.

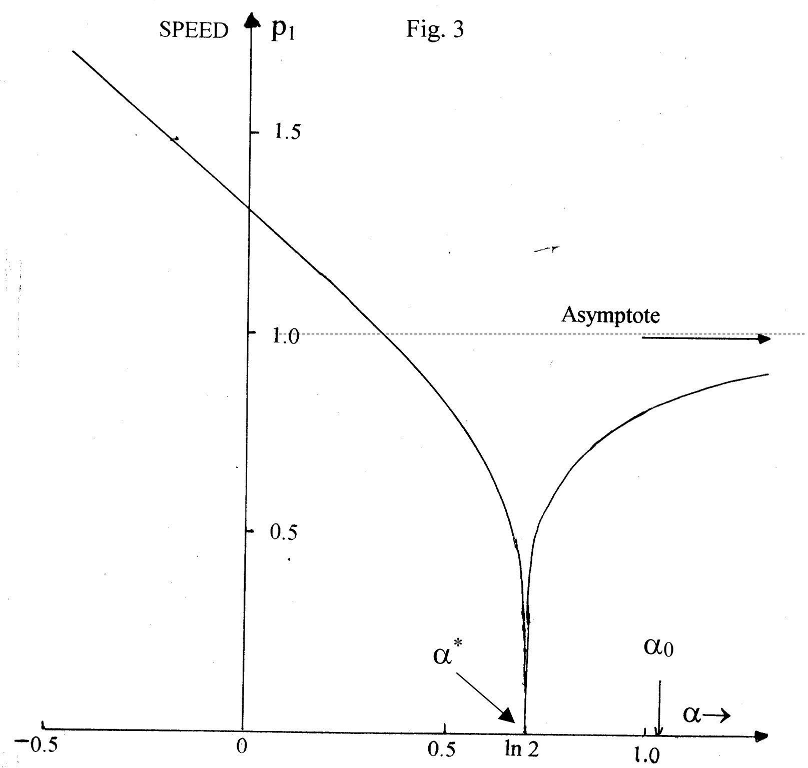


**Fig 3. Variation of $p_1(\alpha)$ with $\alpha$, $-\infty < \alpha < +\infty$, according to the Eqns. (14) and (15), for n = 3. (Source [18])**

Fig. 3 shows the plot of Eqns. (14) and (15) modulus of speed versus $\alpha$. $|p|$ is chosen for simpler visualization of the variation of speed versus $\alpha$ for the region ($\alpha < \alpha^*$). As shown in Fig. 3, as $\alpha$ increases and tends to the branch point at ln2, $p_1$ decreases rapidly to zero. As $\alpha$ crosses the branch point $\alpha^*$ and tends to infinity, $p_1$ increases and attains the asymptotic value 1. The value $\alpha_0$ at which the cusp exists in space-time curve is shown by an arrow.

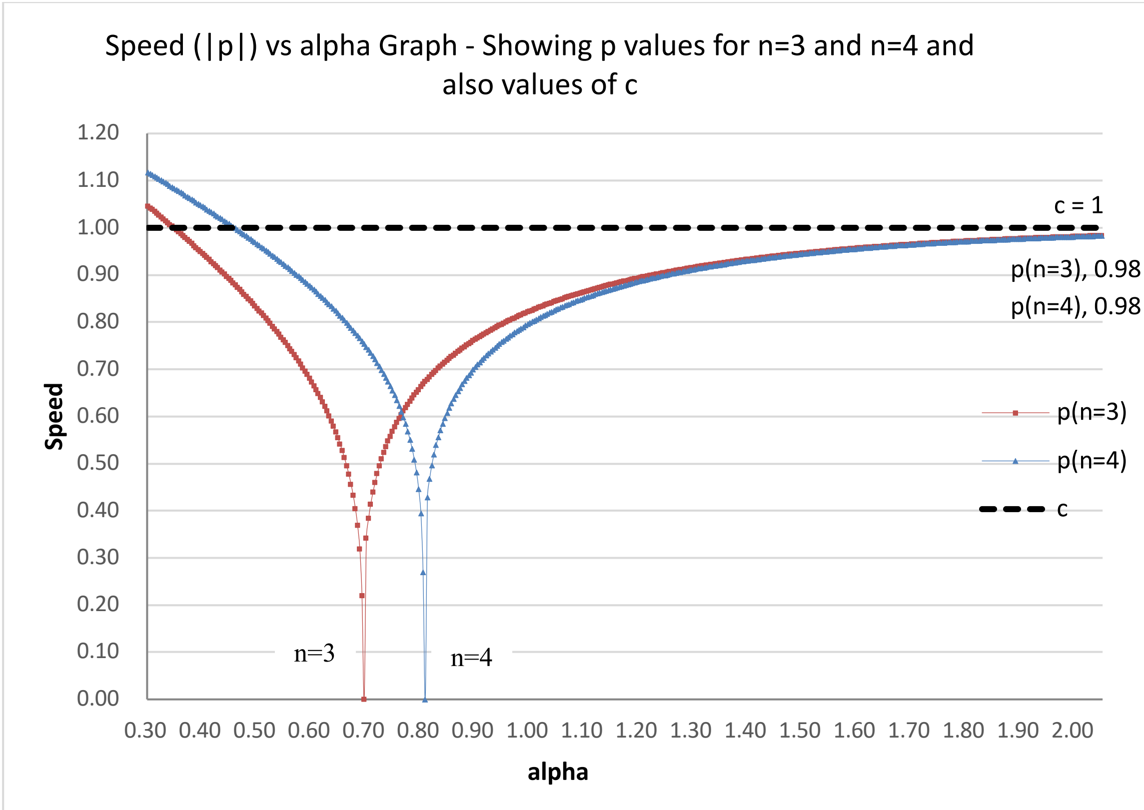


**Fig. 3(a). Variation of $|p_1(\alpha)|$ with $\alpha$, $-\infty < \alpha < +\infty$, for n=3 and n=4**

## Interpreting the Analytical Solution

When $\alpha = \frac{1}{2} \ln (n+1)$, it can be verified that $K_1$ [defined by Eq. (23)] = K [as defined by Eq. (11)], as shown in Appendix A1 (d). Even though $\alpha = \frac{1}{2}\ln (n+1)$ is a singularity of the parametric solution, speed $p_1(\alpha) \to 0$ but $p_1(\alpha) \neq 0$. Therefore, we see that the parametric solution at the branch point $\alpha = \frac{1}{2} \ln (n+1)$, where the speed is zero, tends to the analytical solution.

Thus, Eqns. (9) and (11) appear to correspond to a *Rest-frame of Classical Physics*.

*Thus, so far, the solutions yield results conforming to both Quantum Physics, as well as, Special Theory of Relativity.*

## Various values for Case of n=3

In Eqns. (12) and (13), by differentiating $\psi_1(\alpha)$ and $\xi_1(\alpha)$ with respect to $\alpha$, we find that both attain minimum value at $\alpha = \alpha_0$, given by the roots of the equation

$$\cosh(2\alpha) = (n+1) \tag{24}$$

which for n = 3 has two roots given by:

$$\alpha_0 = \pm 1.0317 \tag{25}$$

Eq. (24) is derived in Appendix A1 (e).

As seen in Fig. 2, the two values of $\alpha_0$ correspond to two cusps of the $\psi_1, \xi_1$ curve, and are located in the positive half space, one each on either side of the line $\xi_1 = 0$.

The product of minimum values of $\psi_1$ and $\xi_1$ is given by

$$\psi_1(\alpha_0)\xi_1(\alpha_0) = +3.9363 \text{ or } -0.615$$

according to $\pm\alpha_0$ because $\xi_1(\alpha_0) < 0$.

Further $p_1(\alpha_0) = 0.8374$ and $p_1(-\alpha_0) = 2.35$; $p_1(0) = 3^{1/4}$, $\xi_1(0) = 0$ and $\psi_1(0) = (1/3)^{3/8}$.

**Interpreting Solutions of Eq. (5)**

In the Eqns. (17) and (18), the variation of $\psi_2$ with $\xi_2$ is shown in Fig. 4 for n=3, where we see that it is a smooth curve which has only 'Future Time' with no 'Past Time'. The variation of $p_2(\beta)$ with $\beta$ is shown in Fig. 5, where we note that $1 < p_2 < \infty$.

Further Eq. (19) shows that

$$p_2(\beta) \to 1 + \exp(-\beta), \ \beta \to \infty, \tag{26}$$

which fulfills the requirement of relativity principle asymptotically as $\beta \to \infty$.

In Fig. 4, when $\psi_2 = 0$ we see that $\xi_2 \neq 0$. In the whole range of $\beta$, $\psi_2$ remains in the half space $\xi_2 > 0$. This can be interpreted as what is currently the accepted view in physics that: *'Time' has no past but only future*. The straight line shows the relativistic asymptote.

Fig. 4

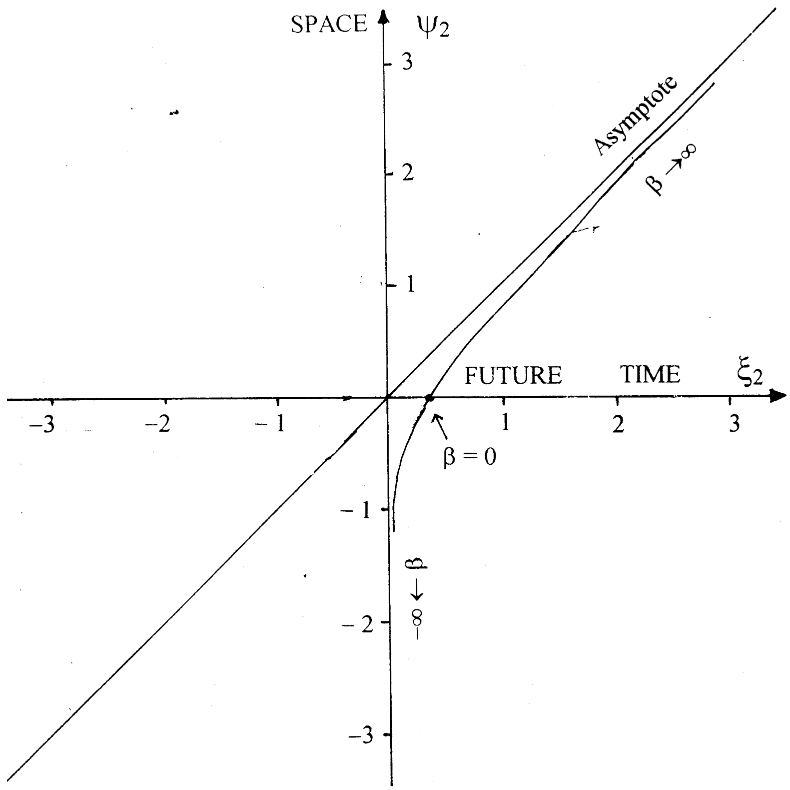


**Fig 4. Variation of $\psi_2$ and $\xi_2$, according the Eqns. (17) and (18) for n=3, with β, where $-\infty < \beta < \infty$. (Source [18])**

There are interesting features of the solutions given by the Eqns. (17) and (18). These can be combined and rewritten in the form

$$\psi_2^{(n+2)} = K_2 \xi_2^{(n)} \qquad (27)$$

where,

$$K_2 = [\sinh^{(n+2)}(\beta)] / [\cosh^n(\beta)] . \qquad (28)$$

Further, we see that $p_2(\beta)$ given by the Eq. (19) vanishes when

$$\exp(\beta) = i\,(n+1)^{1/2}. \qquad (29)$$

where $i = \sqrt{(-1)}$.

As it was shown earlier that $K_1 = K$, now using Eq. (29), it can also be shown that $K_2 = -K$, given by Eq. (11). Therefore, we see that the parametric solution of Eq. (5), which represents 'time-like' space-time, coincides with the analytical solution at an imaginary branch point.

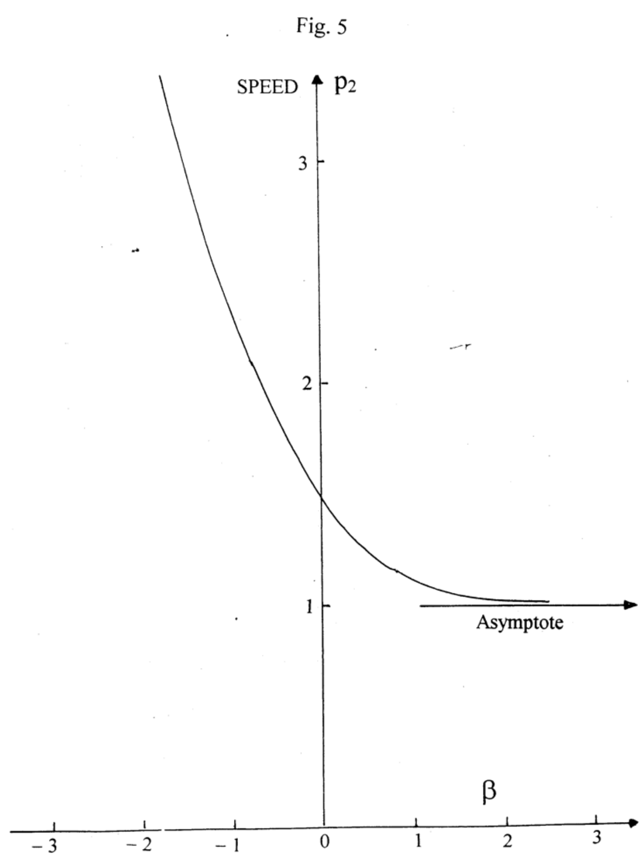


**Fig 5. Variation of $p_2$ with β according to Eq. (19). (Source [18])**

In Fig. 5, it is seen that as β varies from $-\infty < \beta < +\infty$, $p_2$ decreases from infinity approaching unity asymptotically.

The solutions given by this model confirms to TSR that any entity at superluminal speeds remains at speeds greater than speed of light.

It is tempting to interpret $p_2(\beta)$ is a candidate to represent Tachyons, i.e., Faster-than-Light particles, [2]. *Thus, this portion of the Space-Time regime may be named Superluminal Space-Time.* It may also be a candidate to represent the creation of elementary particles, for example, the emission of a photon by an atom, as there is no past for a photon that is just emitted and brought into creation. The apparent origin of a photon is an atom, but the spatial-temporal details of the transitions of atomic energy levels are not accessible to us. The whole space participates in the process of generating a photon, and the time interval for this to happen is of the order of $\xi_2(0)$, which for n=3 is given by, $\xi_2(0) = 1 / \{5\}^{(5/8)} = 0.37$ units of time.

## 6. Scale Factor $\lambda^*$ in the Equation [Eq. (2)]

The mathematical results so far obtained were based on the dimensional considerations of the product of two physical constants, G and h, and therefore, one may be skeptical about their universal applicability. A brief mention was made earlier about the generality of the model. Now, we study four cases to illustrate the different choices of scale factors with the specific goal of answering the question raised in Sec. 2 above: Whether the Model II of space-time includes the Model I.

We recall that $\lambda^*$ in Eq. (2) is a scale factor. This non-dimensional scale factor $\lambda^*$ is like a template which allows different structures to be studied.

For example, we can choose a scale factor in terms of known fundamental constants comprising of *universal constants* such as G and h, and *material constants* such as 'm', the unit of mass, and 'e', the unit of electrical charge.

Each case can be studied separately to understand the resulting property of space-time. Some cases of $\lambda^*$are studied below:

| **Case** | **Scale Factor, λ*** | **Space-Time Equation** | **n** | **Natural Constants** | | **Scaling Length** |
|---|---|---|---|---|---|---|
| | | | | Universal | Material | |
| (a) | $(e^2/hc)(e^2/Gm^2)$ | $(x^2 - c^2 t^2) = (e^2/hc)(e^2/Gm^2)\ Gh/c^3 = (e^4/m^2)(1)(1/c^4)$ | 4 | | e, m | $U_s$, is $1.5\text{x}10^{-18}$ m. Classical radius of proton. |
| (b) | $(e^2/hc)$ | $(x^2 - c^2 t^2) = (e^2/hc)\ Gh/c^3 = (e^2)\ (G)\ (1/c^4)$ | 4 | G | e | $U_s$, is $1.3\text{x}10^{-36}$ m. |
| (c) | $(e^2/Gm^2)$ | $(x^2 - c^2 t^2) = (e^2/Gm^2)\ Gh/c^3 = (e^2/m^2)\ (h)\ (1/c^3)$ | 3 | h | e, m | $U_s$, $4.5\text{x}10^{-17}$ m. In range of particle physics. |
| (d) | 1 | $(x^2 - c^2 t^2) = Gh/c^3 = (1)\ (Gh)\ (1/c^3)$ | 3 | h, G | | $U_s$, is $4.0\text{x}10^{-35}$ m. Planck's length |

**Table 1: Space-Time Structure and Speed-Index (n)**

Table 1 shows the space-time relation resulting from different combinations of scale factor, $\lambda^*$. The $n^{th}$ power of c in the resulting relation is referred to as the Speed-Index. We see that the space-time relation ends up being defined in terms of some universal or material constant.

From Table 1, we see that though we have based our discussion on Planck's length, the model allows the use of other units for length instead of Planck's length, which serve the same purpose of giving the natural units for space-time structure. *Thus, the proposed model is a generic quantization model of the structure of space time.*

In the case (a), the space-time becomes independent of both Planck's constant and also Newtonian Gravitational constant, but is determined by electrostatic charge and inertial mass of material particle. The space-time structure seems to depend on the material properties, as well as, the speed of the particle, more in accordance with the view of Mach than of Kant.

The distinctive feature of the Model II is that the space-time relation is determined by an inhomogeneous differential equation, where Gh acts as a control. This is no doubt a departure from the principle of special relativity which is based on the implicit assumption that space-time is source free.

The merit in the Model II is that in spite of a seeming departure from principle of special relativity, we are able to recover the conditions needed for validity of TSR without making any ad-hoc assumptions. Further, the mass-energy relation which is the contribution of Theory of Special Relativity is not affected by this model which deals with only kinematic variables, and their scale factors.

The above four cases demonstrate that the Model I is contained in the Model II. Hence, Model II, which depends on the product (Gh), becomes a unifying structure of Space-Time for Relativistic, as well as, Quantum Space-Time Structure. Thus, Model II is a generic and a unified model of Space-Time.

## 7. Concept of Quantum Space-Time

We present the concept of Quantum Space-Time, as shown by Eqns. (12) and (13). In Fig. 2, representing the space-time structure for n=3, there may be multiple α values, i.e. multiple space-time coordinates, with the same speed. This means for a given speed there are multiple position values.

The implications of the dashed-line FG in Fig. 2 is explained as follows in Fig. 6 (an expanded visualization of some portion of the space-time relation shown in Fig. 2).

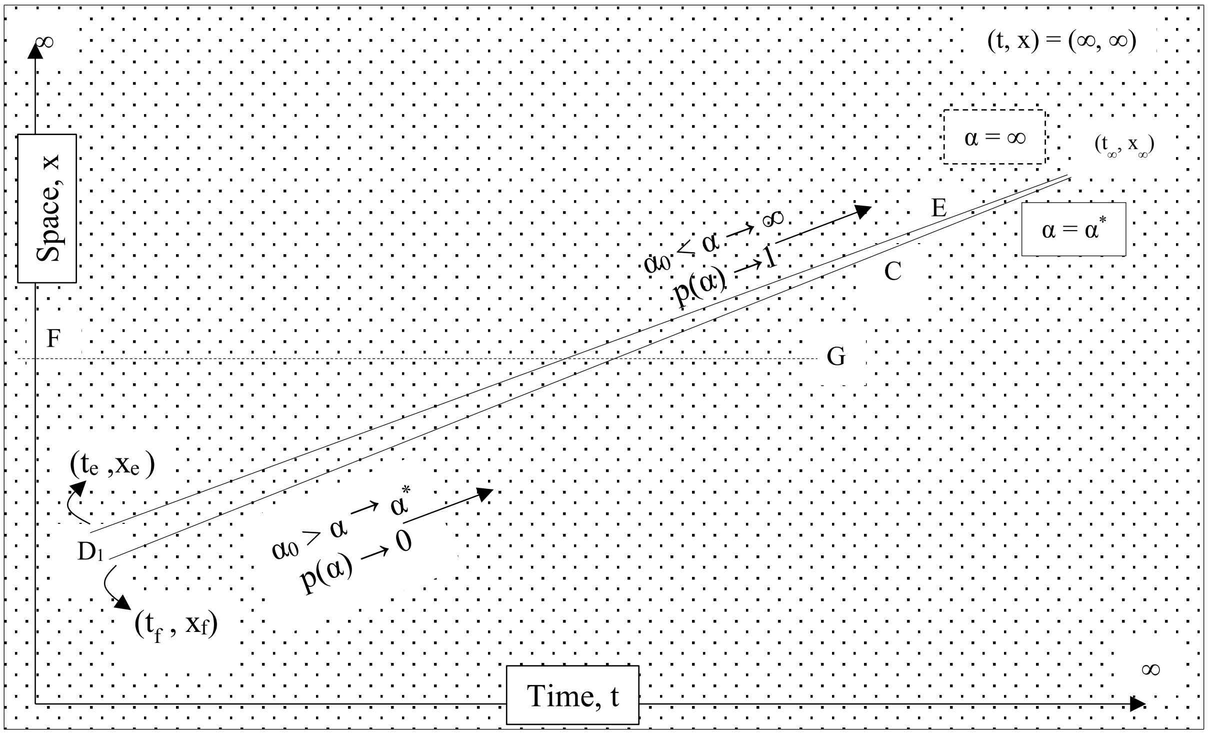


**Fig 6.** Expanded visualization of the Space-Time relation from Fig. 2, highlighting the line FG

In Fig. 6, the segment $CD_1E$ of Fig. 2 is shown. The horizontal dashed-line FG represents a set of coordinates with the same space coordinate. It is seen that for the same position there are two speeds. So, we have multiple positions for a given speed as well as multiple speeds for a given position. This is not consistent with Classical space-time relations.

This multiplicity is pervasive through most of the space-time structure as can be seen from Fig. 2. The relation labelled D2, shown to the left of the axis, $(-\alpha_0 > \alpha > -\infty)$ creates multiple intersection points for any such FG line drawn in this graph for a given location. It is an open question whether the difficulties in interpretation of the quantum events may be related to this feature of space-time.

*Hence, we may interpret the entire space-time relation as being Quantum Space-Time (QST).*

**Regimes of Space-Time at Subluminal speeds**

We see in the subluminal region (physical space-time below in Fig. 7) space-time regimes, which are dependent on α, and therefore dependent on speed. This is shown pictorially in Fig. 7 for a visual understanding of space-time morphing into different regimes, for the case of n=3. This figure is a simplified version of Fig. 3 overlaying the regimes of space-time on the speed-α graph.

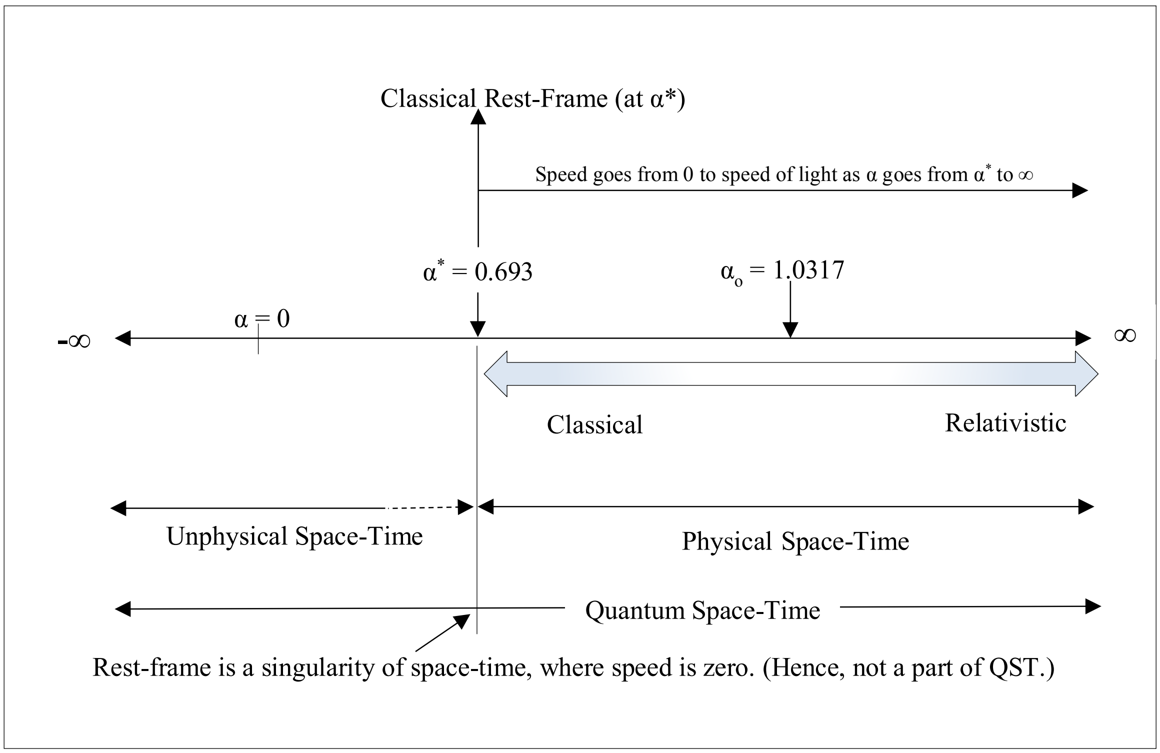


**Fig 7. Simplified representation of Fig. 3, overlaying QST and Space-Time regimes**

Fig. 7 shows the changing characteristics of space-time depending on speed, and therefore, on α. In what follows we examine the interpretation of different values of α and how they map to space-time regimes.

We first present the significance of $\alpha_*$. As discussed earlier in Sec. 4(a), the analytical solution Eq. (9) which passes through the origin, (as shown in Fig. 1) indicating possibility of speeds greater than the speed of light, does not conform to the relativity principle. But we found that the analytical solution coincides with the parametric solution at the branch point, $\alpha^*$ as shown in Fig. 3. Thus, the analytical solution conforms with the relativity principle asymptotically. As the branch point $\alpha^*$ is reached; i.e. speed goes to 0, the space-time curve becomes increasingly flat and $d\psi_1 / d\xi_1$ decreases as $1/ \xi_1^{\{2/ (n+2)\}}$ as $\xi_1 \to\infty$ . This can be seen in Fig. 1. The implication of this result in the analytical solution is that at the branch point space and time are independent of speed, which is consistent with the fundamental assumption of classical physics. It fulfills a necessary condition for stable matter to exist for classical physics. Hence, this portion of the Space-Time regime at $\alpha^*$ may be termed as *Classical regime*.

On the other hand, the function $p_1(\alpha)$, in the range $+\alpha_0 < \alpha < \infty$, appears to be a candidate to represent dynamics of material media, since 'speed' is never larger than that of light in vacuum as required by relativity principle. This portion of the space-time regime may be termed as the *Relativistic regime*.

## 8. Discussion

Our notions of space-time underwent a major revision by the theory of special relativity. New problems arose in the wake of attempts to interpret quantum mechanics in which the role of an "observer" became an indispensable issue [11] and in the understanding of what we mean by the words 'past' and 'future' (See Fig. 2). The parametric solutions of Eq. (4) provide a region of space-time which has 'little past' but has 'entire future' (Fig. 2).

The two physical variables space and time cannot be assumed to be identical, and are ontologically different [19]. There are suggestions that the 'fine structure constant' is slowly increasing over cosmological timescales. One of the suggestions for this increase is that the speed of light is not strictly constant when one considers cosmological time scales [3]. It is hoped that the kinematical considerations of space-time structure throw new light on the pedagogic issues and stimulate further inquiry into the space-time structure on the basis of non-dimensional numbers, which has a long history [5] [9][14].

**Discussion of results for case (n=4)**

The case of n=4 is interesting because it arises from both choices (a) and (b) for $\lambda^*$ mentioned above in Table 1, and the space-time relation is given by Eq. (22), which takes the form

$$\psi_1^6 = + K\,\xi_1^4 \quad , \tag{22a}$$

where, K = 9.1125.

If we take square root of both sides of Eq. (22a), we obtain

$$\psi_1^3 = \pm\, 3.019\,\xi_1^2. \tag{22b}$$

The positive sign in Eq. (22b) gives a frame in which the space-time relation resembles Kepler's law. Unexpectedly finding such a result in this model, lends support to ontological reasoning in understanding the nature of space-time.

Our proposed unifying model reduces to Classical Newtonian Space-Time model when 'h' had no role to play.

Let us re-examine the case (b) chosen for $\lambda^*$, in Sec. 6, Table 1. This choice shows that the gravitational and the electrical interactions would jointly lead to Kepler's law. It was clear from Eq. (22a), that the space-time relation corresponds to Kepler's law, and so Planck's constant alone (as in option (b)) cannot give rise to Kepler's law. The ontologically significance of this merits further observation. The 'inverse square law of distance' property of both gravitational and electrical interactions may be the underlying cause; an area for further exploration. Alfven [1] showed that the rings around the Planet Saturn in the solar system can be explained on the basis of interaction between Saturn's magnetic field with 'dusty' interplanetary plasma cloud after the formation of solar system because the coupling of plasma to the central body is dominated by magnetic field and not by gravitational field alone. Based on these observations, the author is inclined to believe that nature utilizes a combination of gravitational and electromagnetic interactions.

The choice (a) shows that Kepler's law holds also if the unit of length is defined by electric charge, mass and the speed of light. It reminds us of the model of an atom based on the analogy of the planetary system depending on the attraction of positively charged nucleus and negatively charged electron.

We recall that the analytical solution corresponds to the singularity in the parametric solution. This singularity is a state of absolute rest because the 'speed' is zero.

(See Eq. (20)). Yet, there is a non-linear relation, Eq. (22), between space and time implying that speed is non-zero. The conjecture of Klein [15]: "It (TSR) implies that at a given space-time point there are a multitude of frames connected by Lorentz transformations one of them being at rest at the given space-time point", is justified in principle.

The functional relation between space and time passes through the origin of space-time, so this may be interpreted not as a speed but as a cosmological feature. It is somewhat surprising to find that the analytical non-linear space-time relation for n=4 (See Eqns.(22a) and (22b) has an exponent 2/3, which is the same as that obtained in the Einstein-de Sitter cosmological model [10]. This area may be interesting for further exploration.

**Discussion of results for case (n=3)**

The case of n=3 arising from the choice (c) and (d) for $\lambda^*$ in Table 1 is even more interesting. This corresponds to Quantum Physics (as h appears as a factor) and the space-time relation is given by Eq. (22) which takes the form

$$\psi_1^5 = + K\, \xi_1^3 \quad , \qquad (22c)$$

where, K= 7.2338.

If we use the scaling argument, then this corresponds to an inverse-square force law resembling Newton's Gravitational law with an added assumption that the angular momentum be considered a constant, as shown in the Appendix A1 (f) Eq. (A-29), so

$$F \ \alpha \ (1/r^2) \qquad (A\text{-}29)$$

which is the same as Newton's Gravitational law or Coulomb's law.

Thus, for both n=3 and n=4, the space-time structure supports an inverse-square force law.

**The cusps in Space-Time relation**

It may be recalled that Fig. 3 is the parametric relations between 'space' $\psi_1$ ($\alpha$) and 'time' $\xi_1$ ($\alpha$) which are 'scaled variables' defined by Eqns. (12) and (13), assuming $\lambda^*$ = 1 and therefore the value of n = 3. It is seen that space-time relation has two cusps when $\alpha$ satisfies the equation:

$$\cosh(2\alpha) = n+1 \qquad (24)$$

Since n=3, the Eq. 24 gives two values of α viz. $\alpha_0 = \pm 1.0317$. In Fig. 2, the value $\alpha_0 = +1.0317$, which is a cusp, has an interesting effect. As $\alpha^* < \alpha <= \alpha_0$, speed as well as acceleration increases. However, past the cusp, as $\alpha_0 <= \alpha \rightarrow \infty$, speed continues to increase, but acceleration decreases. This indicates a discontinuity in the 'acceleration', such that no entity can exceed the speed of light (as per Eq. (21)). Fig. 3 shows the variation of speed for α, $-\infty < \alpha < +\infty$. It is seen that the speed has an asymptotic limit at the speed of light. Therefore, we see that the requirement of TSR is fulfilled as a mathematical result in this regime.

The cusp at $\alpha_0 = -1.0317$ indicates a 'discontinuity' in 'deceleration' from superluminal to subluminal domain to reach a state of rest as $\alpha \rightarrow \ln 2$.

These discontinuities appear to be like thermodynamic phase changes at a certain temperature. It is a feature that needs more scrutiny which is beyond the scope of this paper. Though the existence of superluminal speeds was not verified, the logic of quanta became the theory of photons and electrons [13]. This theory does not need Newtonian constant, G, and depends only on the Planck's constant, h, and the speed of light, c, because the electronic charge could be expressed as $\sqrt{(hc)}$ apart from a numerical factor.

## 9. Conclusions

We presented a kinematical model of space-time assuming Planck's length $\sqrt{(Gh/c^3)}$ has a role to play in the relation between space and time variables, such as in Minkowski's space-time relation. The space-time model is represented by two first-order fifth- (or higher) degree inhomogeneous differential equations.

**Impact on TSR:**

In our model, Gh acts as a control in the space-time relation, which is a departure from TSR which assumes that the space-time is source free. If Gh is a finite, non-zero constant, the space-time relation becomes very similar to that of TSR, and TSR becomes valid asymptotically at the speed of light. The new model also shows that no entity at rest can be accelerated to speed of light.

Historically, Bohr introduced the concept of quantized angular momentum for electron orbits in an atom. Although it was somewhat of an ad-hoc assumption, it helped explain the atomic spectrum very well. Subsequently, Louis de Broglie postulated a theoretical framework describing the wave-like behavior of matter, which helped validate Bohr's assumptions. Subsequently, the wave-like behavior of matter was experimentally verified, which retrospectively justified the assumptions made by both Bohr and de Broglie.

An arbitrary assumption made in TSR is the constancy of the speed of light. The consequence of this assumption is that matter cannot exceed the speed of light. Our model considers varying speeds all of which are scaled to the speed of light. The mathematical results arising from our work also show that speeds cannot exceed the speed of light in the physical space-time. We see this result as an affirmation of the assumption of TSR, that the speed of light is a constant, which now can be considered as requirement to TSR.

We see this development as being akin to the Bohr, de Broglie development, where a prior ad-hoc assumption is clarified by a later theoretical model. Like in the prior development, we hope that future experimental verification of some of the results from this work solidify the theoretical models of Space-Time structure.

**Impact on Quantum Physics:**

This model has been shown to be a general and unifying model of space-time encompassing classical, relativistic and quantum concepts, which depends on the fundamental constants, G and h. This model shows that the entire space-time structure is Quantum in nature, which is why the structure is referred to as Quantum Space-time (QST) structure,

Quantum mechanics is seen as a direct consequence of the Quantum space-time structure revealed by this model. This model shows that for a given position, speed is indeterminate, and vice-versa. This feature of the QST structure conceptually reconfirms the Uncertainty Principle of Quantum mechanics.

**Regimes in Space-time structure:**

We show that the QST structure may morph between different space-time regimes depending on speed. As speed goes to 0, QST structure reduces to the Classical Newtonian space-time regime. As speed approaches the speed of light, QST structure reduces to the Relativistic space-time regime. Cusps are seen in the space-time structure which would lead to discontinuous transitions in the space-time structure.

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# APPENDICES

## A1. Solutions of Differential Equation Eq. (3):

### (a) Analytical solution of Eq. (4) – Equations (9) and (11)

Assume a solution of the form of a power law, as shown below

$$\psi = A\,\xi^{m} \qquad \text{(A-1)}$$

where A is some constant.

Taking derivative of 'ψ' with respect to 'ξ' in Eq. (A-1), we get

$$p = d\psi/d\xi = A\,m\,\xi^{(m-1)} \qquad \text{(A-2)}$$

Substituting these two equations (A-1) and (A-2) in Eq. (4), we get

$$(Am\xi^{(m-1)})^{n}\,[\,(A\xi^{m})^{2} - (Am\xi^{(m-1)})^{2}.\,\xi^{2}] = 1 \qquad \text{(A-3)}$$

$$A^{(n+2)}\,m^{n}\,[\xi^{(mn-n+2m)} - m^{2}\xi^{(mn - n + 2m - 2 + 2)}\,] = 1$$

$$A^{(n+2)}\,m^{n}\,[\xi^{(mn-n+2m)} - m^{2}\xi^{(mn- n+2m)}\,] = 1$$

$$A^{(n+2)}\,m^{n}\xi^{(mn-n+2m)}[\,1 - m^{2}] = 1$$

This implies that the exponent of ξ must be 0. i.e.

$$mn - n + 2m = 0 \qquad \text{(A-4)}$$

$$m(n+2) = n$$

$$m = n/(n+2) \qquad \text{(A-4a)}$$

and $$A^{(n+2)}[n/(n+2)]^{n}[\,1 - (n/(n+2))^{2}] = 1 \qquad \text{(A-5)}$$

Substituting Eq. (A-4a) and (A-5) in (A-1), we get

$$\psi = A(n)\,\xi^{(n/(n+2))} \qquad \text{(A-1a)}$$

$$\psi^{(n+2)} = A^{(n+2)}(n)\;\xi^{n} = K\,\xi^{n} \qquad \text{(A-s1) = Eq.(9)}$$

where

$$K = [\,(n+2)^{(n+2)}/n^{n}\,.1/\{4(n+1)\}] \qquad \text{(A-s2) = Eq.(11)}$$

**(b) Parametric solution of Eq. (6) – Equation (15)**

$$p(\alpha) = d\psi / d\xi \quad (6)$$

$$= (d\psi/d\alpha) / (d\xi/d\alpha) \quad (A\text{-}6)$$

Now, differentiating ψ with respect to α, we get, from Eq. (12)

$$d\psi/d\alpha = \{ \sinh(\alpha) / p(\alpha)^{n/2}\} + \{ \cosh(\alpha). (-n/2). 1 / p(\alpha)^{(n/2+1)} . p'(\alpha) \} \quad (A\text{-}6a)$$

$$= 1/ p(\alpha)^{n/2}\{ \sinh(\alpha) - n/2 \cosh(\alpha) p'(\alpha)/p(\alpha) \} \quad (A\text{-}6b)$$

Now, differentiating ξ with respect to α, we get, from Eq. (13)

$$d\xi/d\alpha = \{ \cosh(\alpha)/p(\alpha)^{n/2+1} \} + \{ \sinh(\alpha). (-n/2 - 1).1 / p(\alpha)^{(n/2+2)} . p'(\alpha) \} \quad (A\text{-}6c)$$

$$= 1/ p(\alpha)^{(n/2+1)}\{\cosh(\alpha) - (n/2 + 1) \sinh(\alpha) p'(\alpha)/p(\alpha) \} \quad (A\text{-}6d)$$

Now substituting from Eqns. (A-6b) and (A-6d) in Eq. (A-6), we get

$$p(\alpha) = p(\alpha).\{\sinh(\alpha) – n/2 \cosh(\alpha) p'(\alpha)/p(\alpha)\}/\{\cosh(\alpha)-(n/2+1).\sinh(\alpha). p'(\alpha)/p(\alpha)\} \quad (A\text{-}7a)$$

$$1 = \{\sinh(\alpha) - n/2 \cosh(\alpha) p'(\alpha)/p(\alpha)\}/\{\cosh(\alpha) - (n/2 + 1).\sinh(\alpha). p'(\alpha)/p(\alpha)\} \quad (A\text{-}7b)$$

$$\{\sinh(\alpha) - n/2 \cosh(\alpha) p'(\alpha)/p(\alpha)\} = \{\cosh(\alpha) - (n/2 + 1).\sinh(\alpha). p'(\alpha)/p(\alpha)\}$$

Collecting all functions of ‘p’ on one side, we get

$$p'(\alpha)/p(\alpha) = \{\cosh(\alpha) - \sinh(\alpha)\} / \{\sinh(\alpha) - (n/2) (\cosh(\alpha) - \sinh(\alpha))\} \quad (A\text{-}8a)$$

$$p'(\alpha)/p(\alpha) = \{\coth(\alpha) - 1\} / \{1- (n/2) (\coth(\alpha) - 1)\} \quad (A\text{-}8b)$$

which is a differential equation in p and α

Integrating Eq. (A-8b) we get

$$\mathrm{Ln}\, (p(\alpha)) = \int \{\coth(\alpha) - 1\} / \{1- (n/2) (\coth(\alpha) - 1)\}\, d\alpha \quad (A\text{-}9)$$

Now evaluating the integral in Eq. (A-9)

Let

$$\coth(\alpha) – 1 = u \quad (A\text{-}10a)$$

then,

$$- \text{cosech}^2(\alpha)\, d\alpha = du \tag{A-10b}$$

$$- (\coth^2(\alpha) -1)\, d\alpha = du \tag{A-10c}$$

$$- (\coth(\alpha) -1)\, (\coth(\alpha) + 1) d\alpha = du \tag{A-10d}$$

$$- u\, (u + 2)\, d\alpha = du \tag{A-10e}$$

$$d\alpha = - du \,/\, \{u\, (u+2)\} \tag{A-10f}$$

Substituting from Eqns. (A-10a) and (A-10f) in Eq. (A-9), we get

$$\text{Ln}\,(p(\alpha)) = \int u \,/\, \{1- (n/2)\, u\} \,.\, 1 \,/\, \{u\, (u+2)\, \} \,.\, (-du) \tag{A-9a}$$

$$= \int 1 \,/\, \{1- (n/2)\, u\} \,.\, 1 \,/\, \{\, (u+2)\, \} \,.\, (-du)$$

$$\text{Ln}\,(p(\alpha)) = \int 1 \,/\, [\{1- (n/2)\, u\} \,.\{\, (u+2)\}] \,.(- du) \tag{A-9b}$$

$$\text{Ln}\,(p(\alpha)) = -2 \int du \,/\, \{(2- nu)\, .(u+2)\} \tag{A-9c}$$

Now Eq. (A-9c) may be written as partial fractions

$$\text{Ln}\,(p(\alpha)) = \left[- 1/(1+n) \int du \,/\, (u+2) \; - n/(1+n) \int du \,/\, (2 - nu)\, \right] \tag{A-9d}$$

$$= [- 1/(1+n)\, \text{Ln}(u+2) \; - n/(1+n)\, (-1/n)\, \text{Ln}\, (2 - nu)\, ]$$

$$= 1/(1+n)\, \text{Ln}\{(2-nu)/(u+2)\}$$

$$= \text{Ln}\{(2-nu)/(u+2)\}^{1/(1+n)}$$

$$\text{Ln}\,(p(\alpha)) = \text{Ln}\{(2-nu)/(u+2)\}^{1/(1+n)} \tag{A-9e}$$

$$p(\alpha) = \{(2-nu)/(u+2)\}^{1/(1+n)} \tag{A-9f}$$

substituting from Eq. (A-10a) in Eq. (A-9f), we get

$$p(\alpha) = [(2-n\{\coth(\alpha) -1\})/(\{\coth(\alpha) - 1\}+2)]^{1/(1+n)} \tag{A-11}$$

$$= [\{2 \sinh(\alpha) - n \cosh(\alpha) + n \sinh(\alpha)\}/\{\cosh(\alpha) + \sinh(\alpha)\}]^{1/(1+n)}$$

$$p(\alpha) = ([2(e^{\alpha}-e^{-\alpha})/2 - n\, \{(e^{\alpha} + e^{-\alpha})/2 -(e^{\alpha} - e^{-\alpha})/2\}]/[(e^{\alpha} + e^{-\alpha})/2 + (e^{\alpha} - e^{-\alpha})/2])^{1/(1+n)} \tag{A-12}$$

$$p(\alpha) = e^{-\alpha}[e^{\alpha} - e^{-\alpha} - n\, e^{-\alpha}]^{1/(1+n)} \quad \text{(A-13)}$$

$$p(\alpha) = [1 - e^{-2\alpha} - n\, e^{-2\alpha}]^{1/(n+1)} \quad \text{(A-14)}$$

which finally leads to

$$p(\alpha) = [1 - (n+1)\, e^{-2\alpha}]^{1/(n+1)} \quad \text{(A-15)} \equiv \text{Eq. (15)}$$

**(c) alpha in terms of p**

From Eq. (A-14), we get by taking log on both sides

$$(n+1)\, Ln(p) = [1 - (n+1)\, e^{-2\alpha}]$$

$$(n+1)\, e^{-2\alpha} = 1 - (n+1)\, Ln(p)$$

$$e^{-2\alpha} = \{1 - (n+1)\, Ln(p)\}/(n+1)$$

$$-2\alpha = Ln\{1 - (n+1)\, Ln(p)\} - Ln(n+1)$$

$$\alpha = ½\, [Ln(n+1) - Ln\{1 - (n+1)\, Ln(p)\}] \quad \text{(A-16)}$$

**(d) For alpha = ½ ln(n+1), $K_1$ = K as in Eq(23)**

From Eq. (23) we obtain

$$K_1 = [\cosh^{n+2}(\alpha)] / [\sinh^{n}(\alpha)] \quad \text{(23)}$$

Substituting the value $\alpha = ½ \ln(n+1)$, in Eq. (23) above, we get

$$K_1 = [\cosh^{n+2}(½ \ln(n+1))] / [\sinh^{n}(½ \ln(n+1))] \quad \text{(A-17)}$$

But

$$\cosh (½ \ln(n+1)) = \{(n+1)^{½} + (n+1)^{-½}\}/2 \quad \text{(A-17a)}$$

and

$$\sinh (½ \ln(n+1)) = \{(n+1)^{½} - (n+1)^{-½}\}/2 \quad \text{(A-17b)}$$

so,

$$\cosh(½ \ln(n+1)) / \sinh(½ \ln(n+1)) = \{(n+1)^{½} + (n+1)^{-½}\} / \{(n+1)^{½} - (n+1)^{-½}\}$$

$$= (n+2) / n \quad \text{(A-17c)}$$

and

$$\cosh^{2} (½ \ln(n+1)) = [\{(n+1)^{½} + (n+1)^{-½}\}/2]^{2}$$

$$= (n+2)^{2}/\{4(n+1)\} \quad \text{(A-17d)}$$

So, substituting form Eq. (A-17c) and (A-17d) in Eq. (23), we get

$$K_1 = [\cosh^n(\alpha)] / [\sinh^n(\alpha)] * [\cosh^2(\alpha)]$$

$$= [\{(n+2)/n\}^n] [(n+2)^2/\{4(n+1)\}]$$

$$= [(n+2)^{(n+2)} / n^n][1/\{4(n+1)\}]$$

$$= K$$

Therefore

$$K_1 = K \qquad \text{(A-18)}$$

**(e) Derivation of the Eq. (24)**

To calculate the minimum values of $\psi(\alpha)$ and $\xi(\alpha)$ we set their derivatives to 0

$$d\psi/d\alpha = 1/ p(\alpha)^{n/2}\{ \sinh(\alpha) - n/2 \cosh(\alpha) p'(\alpha)/p(\alpha) \} \qquad \text{(A-6b)}$$

$$d\xi/d\alpha = 1/ p(\alpha)^{(n/2+1)}\{\cosh(\alpha) - (n/2 + 1) \sinh(\alpha) p'(\alpha)/p(\alpha) \} \qquad \text{(A-6d)}$$

setting them to 0, we get

$$d\psi/d\alpha = 1/ p(\alpha)^{n/2}\{ \sinh(\alpha) - n/2 \cosh(\alpha) p'(\alpha)/p(\alpha) \} = 0 \qquad \text{(A-19a)}$$

$$d\xi/d\alpha = 1/ p(\alpha)^{(n/2+1)}\{\cosh(\alpha) - (n/2 + 1) \sinh(\alpha) p'(\alpha)/p(\alpha) \} = 0 \qquad \text{(A-20a)}$$

so from (A-19a) we have

$$\{\sinh(\alpha) - n/2 \cosh(\alpha) p'(\alpha)/p(\alpha) \} = 0 \qquad \text{(A-19b)}$$

$$p'(\alpha)/p(\alpha) = 2/n \tanh(\alpha) \qquad \text{(A-19c)}$$

and from (A-20a) we have

$$\{\cosh(\alpha) - (n/2 + 1) \sinh(\alpha) p'(\alpha)/p(\alpha) \} = 0 \qquad \text{(A-20b)}$$

$$p'(\alpha)/p(\alpha) = 2/(n+2) \coth(\alpha) \qquad \text{(A-20c)}$$

so from Eqns. (A-19c) and (A-20c) we get

$$2/n \tanh(\alpha) = 2/(n+2) \coth(\alpha) \qquad \text{(A-21a)}$$

$$(n+2) \tanh(\alpha) = n \coth(\alpha) \qquad \text{(A-21b)}$$

$$(n+2) \sinh^2(\alpha) = n \cosh^2(\alpha) \qquad \text{(A-21c)}$$

$$2 \sinh^2(\alpha) = n \{\cosh^2(\alpha) - \sinh^2(\alpha)\} \qquad \text{(A-21d)}$$

$$2 \sinh^2(\alpha) = n \qquad \text{(A-21e)}$$

$$\cosh(2\alpha) – 1 = n \qquad \text{(A-21f)}$$

$$\cosh(2\alpha) = (n + 1) \qquad \text{(A22)=(24)}$$

so, the functions $\psi(\alpha)$ and $\xi(\alpha)$ attain their minimum when

$$\cosh(2\alpha) = (n + 1) \qquad (24)$$

**(f) Derivation of inverse square force relationship for central forces in n=3 case**

If we use the scaling argument, then even when n=3, the Eq. (9) leads to an inverse-square force law resembling Newton's Gravitational law with an added assumption that the angular momentum be considered a constant, i.e.

$$\psi^{(n+2)} = K\, \xi^{n} \qquad (9)$$

which yields, when n=3

$$\psi^{5} = K\, \xi^{3} \qquad (22c)$$

Now,

Centripetal force:

$$F\ \alpha\ v^2 / r \ \ \ldots \qquad (A\text{-}23)$$

where 'α' is to be read as, Force 'F' is proportional to $v^2 / r$

where velocity, $v = (2\, \pi\, r / T)$ and r is the radius and T is the time period

From (A-23) we have,

$$v\ \alpha\ (r / T) \qquad (A\text{-}24)$$

$$v^2\ \alpha\ (r^2 / T^2) \qquad (A\text{-}25a)$$

$$v^2\ \alpha\ (1/r^3 .\, r^3) . (r^2 / T^2). (1/T .T) \qquad (A\text{-}25b)$$

$$v^2\ \alpha\ (1/r^3) . (r^5 / T^3). (T) \qquad (A\text{-}25c)$$

but from Eq. (22c) above we have

$$r^5/T^3 = K \qquad (A\text{-}26)$$

so from Eqns. (A-25c) and (A-26) we have,

$$v^2\ \alpha\ (T/r^3 .\, K)\ \alpha\ (T/r^3) \qquad (A\text{-}25d)$$

$$v^2\ \alpha\ (\ 1/r^3\ .r/v\ )\ \ \alpha\ \ (\ 1/r^2\ .1/v\ ) \qquad (A\text{-}25)$$

from Eqns. (A-23) and (A-25) we now have

$$F\ \ \alpha\ \ (1/r^2\ .\ 1/v)\ (1/r) \qquad (A\text{-}23a)$$

$$F\ \alpha\ \ (1/r^2\ .\ m/mvr\ ) \qquad (A\text{-}23b)$$

but mvr is the angular momentum L

$$mvr = L \qquad (A\text{-}27)$$

Centripetal force F being a central force, angular momentum is always a constant, so we have

$$mvr = L = C \text{ (const)} \qquad \text{(A=27a)}$$

Now, from Eqns. (A-23b) and (A-27a), we have

$$F \,\alpha\, (1/r^2 \,.\, m/C) \qquad \text{(A-23c)}$$

so

$$F \,\alpha\, 1 / r^2 \qquad \text{(A-28)}$$

This means in any quantum mechanical system (n=3) with constant angular momentum, the force is inversely proportional to square of the distance r and the inverse square law holds.

$$F \,\alpha\, 1 / r^2 \qquad \text{(A-29)}$$

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## Appendix B

### Pictorial Representation of some solutions

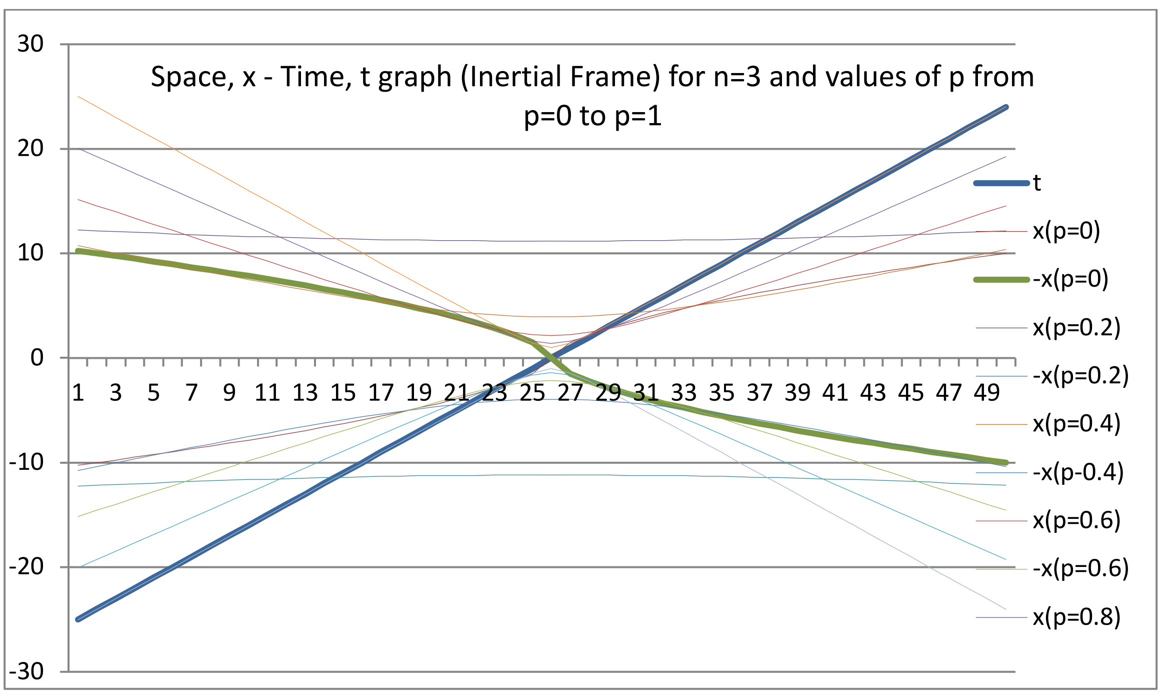


**Fig 8. Parametric Solutions, x-t graph for different values of p**